

FINDING GENERATORS AND RELATIONS FOR GROUPS ACTING ON THE HYPERBOLIC BALL

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ABSTRACT. In order to enumerate the fake projective planes, as announced in [2], we found explicit generators and a presentation for each maximal arithmetic subgroup $\bar{\Gamma}$ of $PU(2, 1)$ for which the (appropriately normalized) covolume equals $1/N$ for some integer $N \geq 1$. Prasad and Yeung [4, 5] had given a list of all such $\bar{\Gamma}$ (up to equivalence).

The generators were found by a computer search which uses the natural action of $PU(2, 1)$ on the unit ball $B(\mathbb{C}^2)$ in \mathbb{C}^2 . Our main results here give criteria which ensure that the computer search has found sufficiently many elements of $\bar{\Gamma}$ to generate $\bar{\Gamma}$, and describes a family of relations amongst the generating set sufficient to give a presentation of $\bar{\Gamma}$.

We give an example illustrating details of how this was done in the case of a particular $\bar{\Gamma}$ (for which $N = 864$). While there are no fake projective planes in this case, we exhibit a torsion-free subgroup Π of index N in $\bar{\Gamma}$, and give some properties of the surface $\Pi \backslash B(\mathbb{C}^2)$.

1. INTRODUCTION

Suppose that P is a fake projective plane. Its Euler-characteristic $\chi(P)$ is 3. The fundamental group $\Pi = \pi_1(P)$ embeds as a cocompact arithmetic lattice subgroup of $PU(2, 1)$, and so acts on the unit ball $B(\mathbb{C}^2) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ in \mathbb{C}^2 , endowed with the hyperbolic metric. Let \mathcal{F} be a fundamental domain for this action. There is a normalization of the hyperbolic volume vol on $B(\mathbb{C}^2)$ and of the Haar measure μ on $PU(2, 1)$ so that $\chi(P) = 3\text{vol}(\mathcal{F}) = 3\mu(PU(2, 1)/\Pi)$. So $\mu(PU(2, 1)/\Pi) = 1$. Let $\bar{\Gamma} \leq PU(2, 1)$ be maximal arithmetic, with $\Pi \leq \bar{\Gamma}$. Then $\mu(PU(2, 1)/\bar{\Gamma}) = 1/N$ and $[\bar{\Gamma} : \Pi] = N$ for some integer $N \geq 1$.

The fundamental group Π of a fake projective plane must also be torsion-free and $\Pi/[\Pi, \Pi]$ must be finite (see [2] for example).

As announced in [2], we have found all groups Π with these properties, up to isomorphism. Our method was to find explicit generators and an explicit presentation for each $\bar{\Gamma}$, so that the question of finding all subgroups Π of $\bar{\Gamma}$ with index N and the additional required properties, as just mentioned, can be studied.

In Section 2, we give results about finding generators and relations for groups Γ acting on quite general metric spaces X . The main theorem gives simple conditions which ensure that a set S of elements of Γ , all of which move a base point 0 by at most a certain distance r_1 , are all the elements of Γ with this property.

In Section 3 we specialize to the case $X = B(\mathbb{C}^2)$, and treat in detail a particular group Γ . This Γ is one of the maximal arithmetic subgroups $\bar{\Gamma} \leq PU(2, 1)$ listed in [4, 5] with covolume of the form $1/N$, N an integer. In this case $N = 864$. Consider the action of Γ on $B(\mathbb{C}^2)$, and let 0 denote the origin in $B(\mathbb{C}^2)$. Two elements, denoted u and v , generate the stabilizer K of 0 in Γ . Another element b of Γ was found by a computer search looking for elements $g \in \Gamma$ for which $d(g.0, 0)$ is small. We use the results of Section 2 to show that the computer search did not miss any such g , and to get a simple presentation of Γ . It turns out that this Γ is one of the Deligne-Mostow groups (see Parker [3]).

We apply this presentation of Γ to exhibit a torsion-free subgroup Π of index 864. The abelianization $\Pi/[\Pi, \Pi]$ is \mathbb{Z}^2 , and so Π is not the fundamental group of a fake-projective plane. However the ball quotient $\Pi \backslash B(\mathbb{C}^2)$ is a compact complex surface with interesting properties, some of which we describe. By a lengthy computer search not discussed here, we showed that any torsion-free subgroup of index 864 in Γ is conjugate to Π , and so no fake projective planes arise in this context.

In Section 4, we calculate the value of r_0 for the example of the previous section.

2. GENERAL RESULTS

Let Γ be a group acting by isometries on a simply-connected geodesic metric space X . Let $S \subset \Gamma$ be a finite symmetric generating set for Γ . Fix a point $0 \in X$, and suppose that $d(0, x)$ is bounded on the set

$$\mathcal{F}_S = \{x \in X : d(0, x) \leq d(g.0, x) \text{ for all } g \in S\}. \quad (2.1)$$

Define

$$r_0 = \sup\{d(0, x) : x \in \mathcal{F}_S\}.$$

Theorem 2.1. *Suppose that there is a number $r_1 > 2r_0$ such that*

- (a) *if $g \in S$, then $d(g.0, 0) \leq r_1$,*
- (b) *if $g, g' \in S$ and $d((gg').0, 0) \leq r_1$, then $gg' \in S$.*

Then $S = \{g \in \Gamma : d(g.0, 0) \leq r_1\}$.

Corollary 2.1. *For Γ, S as in Theorem 2.1, \mathcal{F}_S is equal to the Dirichlet fundamental domain $\mathcal{F} = \{x \in X : d(0, x) \leq d(g.0, x) \text{ for all } g \in \Gamma\}$ of Γ centered at 0.*

Proof. Clearly $\mathcal{F} \subset \mathcal{F}_S$. If $x \in \mathcal{F}_S \setminus \mathcal{F}$, pick a $g \in \Gamma$ so that $d(g.0, x) < d(0, x)$. Now $d(0, x) \leq r_0$ because $x \in \mathcal{F}_S$, and so $d(0, g.0) \leq d(0, x) + d(x, g.0) \leq 2d(0, x) \leq 2r_0 \leq r_1$, so that $g \in S$, by Theorem 2.1. But then $d(0, x) \leq d(g.0, x)$, a contradiction. \square

Corollary 2.2. *Assume Γ, S are as in Theorem 2.1, and that the action of each $g \in \Gamma \setminus \{1\}$ is nontrivial, so that Γ may be regarded as a subgroup of the group $\mathcal{C}(X)$ of continuous maps $X \rightarrow X$. Then Γ is discrete for the compact open topology.*

Proof. Let $V_1 = \{f \in \mathcal{C}(X) : f(0) \in B_{r_1}(0)\}$, where $B_r(x) = \{y \in X : d(x, y) < r\}$. Theorem 2.1 shows that $\Gamma \cap V_1 \subset S$. For each $g \in S \setminus \{1\}$, choose $x_g \in X$ so that $g.x_g \neq x_g$, let $r_g = d(g.x_g, x_g)$, and let $V_g = \{f \in \mathcal{C}(X) : f(x_g) \in B_{r_g}(x_g)\}$. Then $g \notin V_g$. Hence the intersection V of the sets V_g , $g \in S$, is an open neighborhood of 1 in $\mathcal{C}(X)$ such that $\Gamma \cap V = \{1\}$. \square

Before starting the proof of Theorem 2.1, we need some lemmas, which have the same hypotheses as Theorem 2.1.

Lemma 2.1. *The group Γ is generated by $S_0 = \{g \in S : d(g.0, 0) \leq 2r_0\}$.*

Proof. As S is finite, there is a $\delta > 0$ so that $d(g.0, 0) \geq 2r_0 + \delta$ for all $g \in S \setminus S_0$. Let Γ_0 denote the subgroup of Γ generated by S_0 , and assume that $\Gamma_0 \subsetneq \Gamma$. Since Γ is generated by S , there are $g \in S \setminus \Gamma_0$, and we choose such a g with $d(g.0, 0)$ as small as possible. Then $d(g.0, 0) > 2r_0$, since otherwise $g \in S_0 \subset \Gamma_0$. In particular, $g.0 \notin \mathcal{F}_S$. Since $0 \in \mathcal{F}_S$, there is a last point ξ belonging to \mathcal{F}_S on the geodesic from 0 to $g.0$. Choose any point ξ' on that geodesic which is outside \mathcal{F}_S but satisfies $d(\xi, \xi') < \delta/2$. As $\xi' \notin \mathcal{F}_S$, there is an $h \in S$ such that $d(h.0, \xi') < d(0, \xi')$. Hence

$$d(h.0, g.0) \leq d(h.0, \xi') + d(\xi', g.0) < d(0, \xi') + d(\xi', g.0) = d(0, g.0),$$

so that $d((h^{-1}g).0, 0) < d(g.0, 0)$. So $h^{-1}g$ cannot be in $S \setminus \Gamma_0$, by choice of g . Also,

$$d(h.0, 0) \leq d(h.0, \xi') + d(\xi', 0) < 2d(0, \xi') \leq 2(d(0, \xi) + d(\xi, \xi')) < 2r_0 + \delta.$$

Hence $h \in S_0$, by definition of δ . Now $h^{-1}g \in S$ by hypothesis (b) above, since $h^{-1}, g \in S$ and $d((h^{-1}g).0, 0) < d(0, g.0) \leq r_1$. So $h^{-1}g$ must be in Γ_0 . But then $g = h(h^{-1}g) \in \Gamma_0$, contradicting our assumption. \square

Lemma 2.2. *If $x \in X$ and if $d(0, x) \leq r_0 + \epsilon$, where $0 < \epsilon \leq (r_1 - 2r_0)/2$, then there exists $g \in S$ such that $g.x \in \mathcal{F}_S$, and in particular, $d(0, g.x) \leq r_0$.*

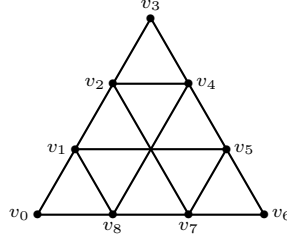
Proof. Since S is finite, we can choose $g \in S$ so that $d(0, g.x)$ is minimal. If $g.x \in \mathcal{F}_S$, there is nothing to prove, and so assume that $g.x \notin \mathcal{F}_S$. There exists $h \in S$ such that $d(h.0, g.x) < d(0, g.x)$, and so $d(0, (h^{-1}g).x) = d(h.0, g.x) < d(0, g.x)$. By the choice of g , $h^{-1}g$ cannot be in S , and also $d(0, g.x) \leq d(0, x)$. So

$$\begin{aligned} d(0, (h^{-1}g).0) &\leq d(0, (h^{-1}g).x) + d((h^{-1}g).x, (h^{-1}g).0) = d(0, (h^{-1}g).x) + d(x, 0) \\ &< d(0, g.x) + d(x, 0) \\ &\leq 2d(0, x) \\ &\leq 2r_0 + 2\epsilon \leq r_1. \end{aligned}$$

But this implies that $h^{-1}g \in S$ by (b) again, since $g, h \in S$. This is a contradiction, and so $g.x$ must be in \mathcal{F}_S . \square

Remark 1. If in Lemma 2.2 we also assume that $\epsilon < \delta/2$, where δ is as in the proof of Lemma 2.1, then the g in the last lemma can be chosen in S_0 . For then $d(0, g.0) \leq d(0, g.x) + d(g.x, g.0) \leq 2d(0, x) \leq 2(r_0 + \epsilon) < 2r_0 + \delta$.

To prove Theorem 2.1, we consider a $g \in \Gamma$ such that $d(g.0, 0) \leq r_1$. By Lemma 2.1, we can write $g = y_1 y_n \cdots y_n$, where $y_i \in S_0$ for each i . Since $1 \in S_0$, we may suppose that n is even, and write $n = 2m$. Moreover, we can assume that m is odd. Form an equilateral triangle Δ in the Euclidean plane with horizontal base whose sides are each divided into m equal segments, marked off by vertices $v_0, v_1, \dots, v_{3m-1}$. We use these to partition Δ into m^2 subtriangles. We illustrate in the case $m = 3$:



We define a continuous function φ from the boundary of Δ to X which maps v_0 to 0 and v_i to $(y_1 \cdots y_i).0$ for $i = 1, \dots, n$, which for $i = 1, \dots, n$ maps the segment $[v_{i-1}, v_i]$ to the geodesic from $\varphi(v_{i-1})$ to $\varphi(v_i)$, and which maps the bottom side of Δ to the geodesic from $g.0 = (y_1 \cdots y_n).0$ to 0.

Because X is simply-connected, we can extend φ to a continuous map $\varphi : \Delta \rightarrow X$, where Δ here refers to the triangle and its interior.

Let $\epsilon > 0$ be as in Lemma 2.2. For a sufficiently large integer r , which we choose to be odd, by partitioning each of the above subtriangles of Δ into r^2 congruent subtriangles, we have $d(\varphi(t), \varphi(t')) < \epsilon$ for all t, t' in the same (smaller) subtriangle.

We now wish to choose elements $x(v) \in \Gamma$ for each vertex v of each of the subtriangles, so that

- (i) $d(x(v).0, \varphi(v)) \leq r_0$ for each v not in the interior of the bottom side of Δ ,
- (ii) $d(x(v).0, \varphi(v)) \leq r_1/2$ for the $v \neq v_0, v_n$ on the bottom side of Δ .

We shall also define elements $y(e)$ for each directed edge e of each of the subtriangles in such a way that

- (iii) $x(w) = x(v)y(e)$ if e is the edge from v to w .

Note that the $y(e)$'s are completely determined by the $x(v)$'s, and the $x(v)$'s are completely determined from the $y(e)$'s provided that one $x(v)$ is specified.

We first choose $x(v)$ for the original vertices $v = v_i$ on the left and right sides of Δ by setting $x(v_0) = 1$ and $x(v_i) = y_1 \cdots y_i$ for $i = 1, \dots, n$. Thus $d(x(v).0, \varphi(v)) = 0 \leq r_0$ for these v 's. Now if $w_0 = v_{i-1}, w_1, \dots, w_r = v_i$ are the $r+1$ equally spaced vertices of the edge from v_{i-1} to v_i , we set $x(w_j) = x(v_{i-1})$ for $1 \leq j < r/2$ and $x(w_j) = x(v_i)$ for $r/2 < j \leq r-1$. Then if $1 \leq j < r/2$, we have

$$\begin{aligned} d(x(w_j).0, \varphi(w_j)) &= d(x(v_{i-1}).0, \varphi(w_j)) = d(\varphi(v_{i-1}), \varphi(w_j)) \\ &\leq \frac{1}{2}d(\varphi(v_{i-1}), \varphi(v_i)) \quad (*) \\ &= \frac{1}{2}d(0, y_i.0) \leq r_0, \end{aligned}$$

where the inequality $(*)$ holds as φ on the segment from v_{i-1} to v_i is the geodesic from $\varphi(v_{i-1})$ to $\varphi(v_i)$. In the same way, $d(x(w_j).0, \varphi(w_j)) \leq r_0$ for $r/2 < j \leq r-1$.

Having chosen $x(v)$ for the v on the left and right sides of Δ , we set $y(e) = x(v)^{-1}x(w)$ if e is the edge from v to w on one of those sides. So of the r edges in the segment from v_{i-1} to v_i , we have $y(e) = 1$ except for the middle edge, for which $y(e) = y_i$.

For the vertices v on the bottom side of Δ , we set $x(v) = 1$ if v is closer to v_0 than to v_n , and set $x(v) = g = y_1 \cdots y_n$ otherwise, and we set $y(e) = x(v)^{-1}x(w)$ if e is the edge from v to w . Since rm is odd, there is no middle vertex on the side, so there is no ambiguity in the definition of $x(v)$, but there is a middle edge e , and $y(e) = g$ if e is directed from left to right. For the other edges e' on the bottom side, $y(e') = 1$. If v is a vertex of the bottom side of Δ closer to v_0 than to v_n , then

$$d(x(v).0, \varphi(v)) = d(0, \varphi(v)) \leq \frac{1}{2}d(0, g.0) \leq \frac{r_1}{2},$$

since φ on the bottom side is just the geodesic from 0 to $g.0$. Similarly, if v is a vertex of the bottom side of Δ closer to v_n , then again $d(x(v).0, \varphi(v)) \leq r_1/2$.

We now choose $x(v)$ for vertices which are not on the sides of Δ as follows. We successively define $x(v)$ as we move from left to right on a horizontal line. Suppose that v is a vertex for which $x(v)$ has not yet been chosen, but for which $x(v')$ has been chosen for the vertex v' immediately to the left of v . Now $d(\varphi(v), \varphi(v')) \leq \epsilon$ and $d(x(v').0, \varphi(v')) \leq r_0$, so that $d(\varphi(v), x(v').0) \leq r_0 + \epsilon$. By Lemma 2.2, applied to $x = x(v')^{-1}\varphi(v)$, there is a $g \in S$ so that $d(gx(v')^{-1}\varphi(v), 0) \leq r_0$. So we set $x(v) = x(v')g^{-1}$. If e is the edge from v' to v , we set $y(e) = g^{-1} \in S$.

We have now defined $x(v)$ for each vertex of the partitioned triangle Δ so that (i) and (ii) hold, and these determine $y(e)$ for each directed edge e so that (iii) holds. We have seen that $y(e) \in S$ if e is an edge on the left or right side of Δ or if e is a horizontal edge not lying on the bottom side of Δ whose right hand endpoint does not lie on the right side of Δ . Also, $y(e) = 1 \in S$ for all edges e in the bottom side of Δ except the middle one, for which $y(e) = g$. The theorem will be proved once we check that $y(e) \in S$ for each edge e .

Lemma 2.3. *Suppose that v, v' and v'' are the vertices of a subtriangle in Δ , and that e, e' and e'' are the edges from v to v' , v' to v'' , and v to v'' , respectively. Suppose that $y(e)$ and $y(e')$ are in S . Then $y(e'') \in S$ too.*

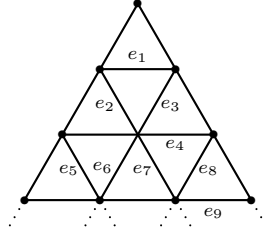
Proof. We have $y(e) = x(v)^{-1}x(v')$, $y(e') = x(v')^{-1}x(v'')$ and $y(e'') = x(v)^{-1}x(v'')$, and so $y(e'') = y(e)y(e')$. If e'' is the middle edge of the bottom side of Δ , then $d(y(e'').0, 0) = d(g.0, 0) \leq r_1$ by hypothesis, and so $y(e'') \in S$ by (b) above. If e'' is any other edge of the bottom side of Δ , then $y(e'') = 1 \in S$. So we may assume that e'' is not on the bottom side of Δ . Hence at most one of v and v'' lies on that

bottom side. Hence

$$\begin{aligned}
d(0, y(e'').0) &= d(x(v).0, x(v)y(e'').0) \\
&= d(x(v).0, x(v'').0) \\
&\leq d(x(v).0, \varphi(v)) + d(\varphi(v), \varphi(v'')) + d(\varphi(v''), x(v'').0) \\
&< d(x(v).0, \varphi(v)) + d(\varphi(v''), x(v'').0) + \epsilon \\
&\leq r_0 + r_1/2 + \epsilon \quad (\dagger) \\
&\leq r_1,
\end{aligned}$$

where the inequality (\dagger) holds because at least one of $d(x(v).0, \varphi(v)) \leq r_0$ and $d(\varphi(v''), x(v'').0) \leq r_0$ holds, since at most one of v and v'' is a vertex of the bottom side of Δ , and if say v is such a point, then we still have $d(x(v).0, \varphi(v)) \leq r_1/2$. \square

Conclusion of the proof of Theorem 2.1. We must show that $y(e) \in S$ for all edges. We use Lemma 2.3, working down from the top of Δ , and moving from left to right. So in the order indicated in the next diagram we work down Δ , finding that $y(e) \in S$ in each case, until we get to the lowest row of triangles.



Then working from the left and from the right, we find that $y(e) \in S$ for all the diagonal edges in the lowest row.

Finally, we get to the middle triangle in the lowest row. For the diagonal edges e, e' of that triangle, we have found that $y(e), y(e') \in S$, and so $y(e'') \in S$ for the horizontal edge e'' of that triangle too, by Lemma 2.3. \square

If we make the extra assumption that the set of values $d(g.0, 0)$, $g \in \Gamma$, is discrete, the following result is a consequence of [1, Theorem I.8.10] (applied to the open set $U = \{x \in X : d(x, 0) < r_0 + \epsilon\}$ for $\epsilon > 0$ small). We shall sketch a proof along the lines of that of Theorem 2.1 which does not make that extra assumption.

Theorem 2.2. *With the hypotheses of Theorem 2.1, let $S_0 = \{g \in S : d(g.0, 0) \leq 2r_0\}$, as before. Then the set of generators S_0 , and the relations $g_1 g_2 g_3 = 1$, where the g_i are each in S_0 , give a presentation of Γ .*

Proof. Let \tilde{S}_0 be a set with a bijection $f : \tilde{s} \mapsto s$ from \tilde{S}_0 to S_0 . Let F be the free group on \tilde{S}_0 , and denote also by f the induced homomorphism $F \rightarrow \Gamma$. Then f is surjective by Lemma 2.1. Let $\tilde{y}_1, \dots, \tilde{y}_n \in \tilde{S}_0$, and suppose that $\tilde{g} = \tilde{y}_1 \cdots \tilde{y}_n \in F$ is in the kernel of f . We must show that \tilde{g} is in the normal closure H of the set of elements of F of the form $\tilde{s}_1 \tilde{s}_2 \tilde{s}_3$, where $\tilde{s}_i \in \tilde{S}_0$ for each i , and $s_1 s_2 s_3 = 1$ in Γ . As $1 \in S_0$, we may assume that n is a multiple $3m$ of 3, and form a triangle Δ partitioned into m^2 congruent subtriangles, as in the proof of Theorem 2.1. The vertices are again denoted v_i , and we write $v_{3m} = v_0$, and $y_i = f(\tilde{y}_i)$ for each i .

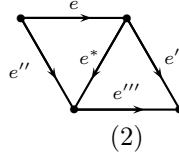
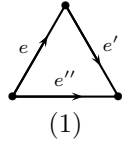
We again define a continuous function φ from the boundary of Δ to X which maps v_0 to 0 and v_i to $(y_1 \cdots y_i).0$ for $i = 1, \dots, 3m$, and which for $i = 1, \dots, 3m$ maps the segment $[v_{i-1}, v_i]$ to the geodesic from $\varphi(v_{i-1})$ to $\varphi(v_i)$. From $y_1 y_2 \cdots y_n = 1$ we see that $\varphi(v_0) = 1 = g = \varphi(v_{3m})$. This time the bottom side of Δ is mapped in the same way as the other two sides.

Let $\epsilon > 0$, with $\epsilon < \delta/2, (r_1 - 2r_0)/2$, and as before, we partition Δ into subtriangles so that whenever t, t' are in the same subtriangle, $d(\varphi(t), \varphi(t')) < \epsilon$ holds.

Using Lemma 2.2 and the remark after it, we can again choose elements $x(v) \in \Gamma$ for each vertex v of each of the subtriangles, so that $d(x(v).0, \varphi(v)) \leq r_0$ for each v , without the complications about the bottom side of Δ which had to be dealt with in the proof of Theorem 2.1. If e is the edge from v' to v , where v' is immediately to the left of v , and v is not on the right side of Δ , then $y(e) = x(v')^{-1}x(v) \in S_0$. We then define elements $y(e)$ for each directed edge e of each of the subtriangles so that $x(w) = x(v)y(e)$ if e is the edge from v to w . Again using Lemma 2.3, with S there replaced by S_0 , and arguing as in the conclusion of the proof of Theorem 2.1, we deduce that $y(e) \in S_0$ for each edge e .

Of the r edges e in the segment from v_{i-1} to v_i , we have $y(e) = 1$ except for the middle edge, for which $y(e) = y_i$. So $y(e_1)y(e_2) \cdots y(e_{3mr}) = 1$, where e_1, \dots, e_{3mr} are the successive edges as we traverse the sides of Δ in a clockwise direction starting at v_0 , and $\tilde{g} = \tilde{y}(e_1) \cdots \tilde{y}(e_{3mr})$.

It is now easy to see that $\tilde{g} \in H$, by using $y(e'') = y(e)y(e')$ in the situation of (1), so that $\tilde{y}(e'') = \tilde{y}(e)\tilde{y}(e')h$ for some $h \in H$. Then in the situation of (2), for example, $y(e)y(e') = y(e)(y(e^*)y(e''')) = (y(e)y(e^*))y(e''') = y(e'')y(e''')$,



so that $\tilde{y}(e)\tilde{y}(e') = \tilde{y}(e)(\tilde{y}(e^*)\tilde{y}(e'''))h = (\tilde{y}(e)\tilde{y}(e^*))\tilde{y}(e''')h = (\tilde{y}(e'')h')\tilde{y}(e''')h = \tilde{y}(e'')\tilde{y}(e''')h''$ for some $h, h', h'' \in H$. We can, for example, successively use this device to remove the right hand strip of triangles from Δ , reducing the size of the triangle being treated, and then repeat this process. \square

In the next proposition, we use [1, Theorem I.8.10] to show that under extra hypotheses (satisfied by the example in Section 3), we can omit the $g \in S_0$ for which $d(g.0, 0) = 2r_0$, and still get a presentation.

Proposition 2.1. *Let $X = B(\mathbb{C}^2)$, and suppose that the set of values $d(g.0, 0)$, $g \in \Gamma$, is discrete. Assume also the hypotheses of Theorem 2.1. Then the set $S_0^* = \{g \in S : d(g.0, 0) < 2r_0\}$ is a set of generators of Γ , and the relations $g_1g_2g_3 = 1$, where the g_i are each in S_0^* , give a presentation of Γ .*

Proof. For each $g \in S$ such that $d(g.0, 0) = 2r_0$, let m be the midpoint of the geodesic from 0 to $g.0$. Let M be the set of these midpoints. Let $\delta_1 > 0$ be so small that $2r_0 + 2\delta_1 < r_1$. Since M and S are finite, we can choose a positive $\delta < \delta_1$ so that if $m, m' \in M$ and $g \in S$, and if $d(g.m, m') < 2\delta$, then $g.m = m'$. So if $\gamma, \gamma' \in \Gamma$, $m, m' \in M$ and $B(\gamma.m, \delta) \cap B(\gamma'.m', \delta) \neq \emptyset$, then $d(\gamma.0, \gamma'.0) < 2r_0 + 2\delta < r_1$, so that $\gamma^{-1}\gamma' \in S$ by Theorem 2.1, and $d(m, \gamma^{-1}\gamma'.m') < 2\delta$, so that $\gamma.m = \gamma'.m'$ by choice of δ . Thus if Y is the union of the Γ -orbits of all $m \in M$, then the balls $B(y, \delta)$, $y \in Y$, are pairwise disjoint. Now let X' denote the subset of X obtained by removing all these balls. It follows (because the ambient dimension is > 2) that X' is still simply connected.

Let $U = \{x \in X' : d(x, 0) < r_0 + \delta'\}$ of X' , for some $\delta' > 0$. The proposition will follow from [1, Theorem I.8.10], applied to this U , once we show that if $\delta' > 0$ is small enough, then any $g \in \Gamma$ such that $g(U) \cap U \neq \emptyset$ must satisfy $d(g.0, 0) < 2r_0$. Clearly $d(g.0, 0) < 2r_0 + 2\delta'$ holds, and because of the discreteness hypothesis, if $\delta' > 0$ is small enough, one even has $d(g.0, 0) \leq 2r_0$. Now suppose $d(g.0, 0) = 2r_0$, let m be the midpoint on the geodesic from 0 to $g.0$, and suppose $x \in g(U) \cap U$. Then $d(0, x) < r_0 + \delta'$ and $d(g.0, x) < r_0 + \delta'$. Using the CAT(0) property of X ,

this shows that $d(x, m)^2 + r_0^2 \leq (r_0 + \delta')^2$, so that

$$d(x, m) \leq \sqrt{(r_0 + \delta')^2 - r_0^2},$$

and this last can be made less than δ if δ' is chosen small enough. This contradicts the hypothesis that $x \in U \subset X'$. \square

The fact that X' is simply connected could also be used to modify the version of the proof going through the triangle-shaped simplicial complex.

3. AN EXAMPLE

Let $\ell = \mathbb{Q}(\zeta)$, where ζ is a primitive 12-th root of unity. Then $\zeta^4 = \zeta^2 - 1$, so that ℓ is a degree 4 extension of \mathbb{Q} . Let $r = \zeta + \zeta^{-1}$ and $k = \mathbb{Q}(r)$. Then r and ζ^3 are square roots of 3 and -1 , respectively, and if $\zeta = e^{2\pi i/12}$ then $r = +\sqrt{3}$ and $\zeta^3 = i$. Let

$$F = \begin{pmatrix} -r-1 & 1 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

and form the group

$$\Gamma = \{g \in M_{3 \times 3}(\mathbb{Z}[\zeta]) : g^* F g = F\} / \{\zeta^\nu I : \nu = 0, 1, \dots, 11\}.$$

We shall use the results of Section 2 to find a presentation for Γ . Let us first motivate the choice of this example. Now $\iota(g) = F^{-1}g^*F$ defines an involution of the second kind on the simple algebra $M_{3 \times 3}(\ell)$. We can define an algebraic group G over k so that

$$G(k) = \{g \in M_{3 \times 3}(\ell) : \iota(g)g = I \text{ and } \det(g) = 1\}.$$

For the corresponding adjoint group \overline{G} ,

$$\overline{G}(k) = \{g \in M_{3 \times 3}(\ell) : \iota(g)g = I\} / \{tI : t \in \ell \text{ and } \bar{t}t = 1\}.$$

Now k has two archimedean places v_+ and v_- , corresponding to the two embeddings $k \hookrightarrow \mathbb{R}$ mapping r to $+\sqrt{3}$ and $-\sqrt{3}$, respectively. The eigenvalues of F are 1 and $-r \pm \sqrt{2}$. So the form F is definite for v_- but not for v_+ . Hence

$$\overline{G}(k_{v_-}) \cong PU(3) \quad \text{and} \quad \overline{G}(k_{v_+}) \cong PU(2, 1).$$

Letting V_f denote the set of non-archimedean places of k , if $v \in V_f$, then either $\overline{G}(k_v) \cong PGL(3, k_v)$ if v splits in ℓ , or $\overline{G}(k_v) \cong PU_F(3, k_v(i))$ if v does not split in ℓ . With a suitable choice of maximal parahorics \overline{P}_v in $\overline{G}(k_v)$, the following group

$$\bar{\Gamma} = \overline{G}(k) \cap \prod_{v \in V_f} \overline{P}_v$$

is one of the maximal arithmetic subgroups of $PU(2, 1)$ whose covolume has the form $1/N$, N an integer. Prasad and Yeung [4, 5] have described all such subgroups, up to k -equivalence. In this case $N = 864$. As in [2], lattices can be used to describe concretely maximal parahorics. We can take $\overline{P}_v = \{g \in \overline{G}(k_v) : g.x_v = x_v\}$, where, in the cases when v splits in ℓ , x_v is the homothety class of the \mathfrak{o}_v -lattice $\mathfrak{o}_v^3 \subset k_v^3$, where \mathfrak{o}_v is the valuation ring of k_v . When v does not split, x_v is the lattice \mathfrak{o}_v^3 , where now \mathfrak{o}_v is the valuation ring of $k_v(i)$. With this particular choice of parahorics, $\bar{\Gamma}$ is just Γ .

The action of Γ on the unit ball $X = B(\mathbb{C}^2)$ in \mathbb{C}^2 is described as follows, making explicit the isomorphism $\overline{G}(k_{v_+}) \cong PU(2, 1)$. Let

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_{\text{diag}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-r \end{pmatrix}, \quad \text{and} \quad F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.2)$$

Then $\gamma_0^t F_{\text{diag}} \gamma_0 = (1-r)F$, so a matrix g is unitary with respect to F if and only if $g' = \gamma_0 g \gamma_0^{-1}$ is unitary with respect to F_{diag} . Now let D be the diagonal matrix with diagonal entries 1, 1 and $\sqrt{\sqrt{3}-1}$. Taking $r = +\sqrt{3}$, if g' is unitary with respect to F_{diag} , then $\tilde{g} = Dg'D^{-1}$ is unitary with respect to F_0 ; that is, $\tilde{g} \in U(2, 1)$. If $Z = \{\zeta^\nu I : \nu = 0, \dots, 11\}$, for an element gZ of Γ , the action of gZ on $B(\mathbb{C}^2)$ is given by the usual action of \tilde{g} . That is,

$$(gZ).(z, w) = (z', w') \quad \text{if} \quad \tilde{g} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} z' \\ w' \\ 1 \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{C}.$$

Now let u and v be the matrices

$$u = \begin{pmatrix} \zeta^3 + \zeta^2 - \zeta & 1 - \zeta & 0 \\ \zeta^3 + \zeta^2 - 1 & \zeta - \zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \zeta^3 & 0 & 0 \\ \zeta^3 + \zeta^2 - \zeta - 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. They have entries in $\mathbb{Z}[\zeta]$, are unitary with respect to F , and satisfy

$$u^3 = I, \quad v^4 = I, \quad \text{and} \quad (uv)^2 = (vu)^2.$$

They (more precisely, uZ and vZ) generate a subgroup K of Γ of order 288. Magma shows that an abstract group with presentation $\langle u, v : u^3 = v^4 = 1, (uv)^2 = (vu)^2 \rangle$ has order 288, and so K has this presentation.

Let us write simply 0 for the origin $(0, 0) \in B(\mathbb{C}^2)$, and $g.0$ in place of $(gZ).(0, 0)$.

Lemma 3.1. *For the action of Γ on X , K is the stabilizer of 0.*

Proof. It is easy to see that $gZ \in \Gamma$ fixes 0 if and only if gZ has a matrix representative

$$\begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for suitable $g_{ij} \in \mathbb{Z}[\zeta]$. Since the 2×2 block matrix in the upper left of F is definite when $r = +\sqrt{3}$, it is now routine to determine all such g_{ij} . \square

The next step is to find $g \in \Gamma \setminus K$ for which $d(g.0, 0)$ is small, where d is the hyperbolic metric on $B(\mathbb{C}^2)$. Now

$$\cosh^2(d(z, w)) = \frac{|1 - \langle z, w \rangle|^2}{(1 - |z|^2)(1 - |w|^2)}, \quad (3.3)$$

(see [1, Page 310] for example) where $\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$ and $|z| = \sqrt{|z_1|^2 + |z_2|^2}$ for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $B(\mathbb{C}^2)$.

In particular, writing 0 for the origin in $B(\mathbb{C}^2)$, and using $g.0 = (g_{13}/g_{33}, g_{23}/g_{33})$ and $|g_{13}|^2 + |g_{23}|^2 = |g_{33}|^2 - 1$ for $g = (g_{ij}) \in U(2, 1)$, we see that

$$\cosh^2(d(0, g.0)) = |g_{33}|^2 \quad (3.4)$$

for $g \in U(2, 1)$. Notice that for $gZ \in \Gamma$, the $(3, 3)$ -entry of g is equal to the $(3, 3)$ -entry of the $\tilde{g} \in U(2, 1)$ defined above, and so (3.4) holds also for $gZ \in \Gamma$.

The matrix

$$b = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3 - \zeta^2 + 2\zeta + 2 & \zeta^3 + \zeta^2 - \zeta - 1 & -\zeta^3 - \zeta^2 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & -\zeta^3 + \zeta + 1 \end{pmatrix}$$

is unitary with respect to F . We shall see below that u, v and b generate Γ , and use the results of Section 2 to show that some relations they satisfy give a presentation of Γ . This b was found by a computer search for $g \in \Gamma \setminus K$ for which $d(g.0, 0)$ is small.

Notice that $d(g.0, 0)$ is constant on each double coset KgK . Calculations using (3.4) showed that amongst the 288 elements $g \in \Gamma$ of the form bkb , $k \in K$,

there are ten different values of $d(g, 0, 0)$. Representatives γ_j of the 20 double cosets KgK in which the $g \in bKb$ lie were chosen. The smallest few $|(\gamma_j)_{33}|^2$ and the corresponding $d(\gamma_j, 0, 0)$ (rounded to 4 decimal places) are as follows:

| j | γ_j | $ (\gamma_j)_{33} ^2$ | $d(\gamma_j, 0, 0)$ |
|-----|------------------------|-----------------------|---------------------|
| 1 | 1 | 1 | 0 |
| 2 | b | $\sqrt{3} + 2$ | 1.2767 |
| 3 | $bu^{-1}b$ | $2\sqrt{3} + 4$ | 1.6629 |
| 4 | $bu^{-1}v^{-1}u^{-1}b$ | $3\sqrt{3} + 6$ | 1.8778 |

We use Theorem 2.1 to show that our computer search has not missed any $g \in \Gamma$ for which $d(g, 0, 0)$ is small. Let $S = K \cup K\gamma_2K \cup K\gamma_3K$, consisting of the $g \in \Gamma$ found by the computer search to satisfy $|g_{33}|^2 \leq 2\sqrt{3} + 4$. To verify that S generates Γ , we need to make a numerical estimate. A somewhat longer direct proof that $\langle S \rangle = \Gamma$ is given in the next section.

Proposition 3.1. *For the given $S \subset \Gamma$, the normalized hyperbolic volume $\text{vol}(\mathcal{F}_S)$ of $\mathcal{F}_S = \{x \in B(\mathbb{C}^2) : d(x, 0) \leq d(x, g, 0) \text{ for all } g \in S\}$ satisfies $\text{vol}(\mathcal{F}_S) < 2/864$. The set S generates Γ .*

Proof. Standard numerical integration methods show that (up to several decimal place accuracy) $\text{vol}(\mathcal{F}_S)$ equals $1/864$, but all we need is that $\text{vol}(\mathcal{F}_S) < 2/864$. Let us make explicit the normalization of hyperbolic volume element in $B(\mathbb{C}^2)$ which makes the formula $\chi(\Gamma \backslash B(\mathbb{C}^2)) = 3\text{vol}(\mathcal{F})$ true. For $z \in B(\mathbb{C}^2)$, write

$$t = \tanh^{-1} |z| = \frac{1}{2} \log \left(\frac{1 + |z|}{1 - |z|} \right) = d(0, z),$$

$$\Theta = z/|z| \in S^1(\mathbb{C}^2),$$

$$d\Theta = \text{the usual measure on } S^1(\mathbb{C}^2), \text{ having total volume } 2\pi^2,$$

$$d\text{vol}(z) = \frac{2}{\pi^2} \sinh^3(t) \cosh(t) dt d\Theta.$$

Let $\Gamma' = \langle S \rangle$. Then the Dirichlet fundamental domains \mathcal{F} and \mathcal{F}' of Γ and Γ' satisfy $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{F}_S$. By [4, §8.2, the \mathcal{C}_{11} case], $\text{vol}(\mathcal{F}) = 1/864$. Let $M = [\Gamma : \Gamma']$. Then $\text{vol}(\mathcal{F}') = M\text{vol}(\mathcal{F})$ (and $\text{vol}(\mathcal{F}') = \infty$ if $M = \infty$), so that $M\text{vol}(\mathcal{F}) = \text{vol}(\mathcal{F}') \leq \text{vol}(\mathcal{F}_S) < 2\text{vol}(\mathcal{F})$ implies that $M = 1$ and $\Gamma' = \Gamma$. \square

Proposition 3.2. *For the given $S \subset \Gamma$, the value of $r_0 = \sup\{d(x, 0) : x \in \mathcal{F}_S\}$ is $\frac{1}{2}d(\gamma_3, 0, 0) = \frac{1}{2} \cosh^{-1}(\sqrt{3} + 1)$.*

Proof. We defer the proof of this result to the next section. \square

One may verify that if $g, g' \in S$ and $gg' \notin S$, then $|(\gamma_j)_{33}|^2 \geq 3\sqrt{3} + 6$. So if r_1 satisfies $2\sqrt{3} + 4 < \cosh^2(r_1) < 3\sqrt{3} + 6$, then S satisfies the conditions of Theorem 2.1.

Remark 2. By Corollary 2.1, $\mathcal{F}_S = \mathcal{F}$, so that $\text{vol}(\mathcal{F}_S)$ equals $1/864$.

Having verified all the conditions of Theorem 2.1, we now know that S contains all elements $g \in \Gamma$ satisfying $d(g, 0, 0) \leq r_1$. The double cosets $K\gamma_jK$, $j = 1, 2, 3$, are symmetric because $(buvu)^2v = \zeta^{-1}I$, and $(bu^{-1})^4 = I$ (see below). The sizes of these $K\gamma_jK$ are 288 , $288^2/4$ and $288^2/3$, respectively, adding up to $48,672$, because $\{k \in K : \gamma_2^{-1}k\gamma_2 \in K\} = \langle v \rangle$, and $\{k \in K : \gamma_3^{-1}k\gamma_3 \in K\} = \langle u \rangle$. Proposition 2.1 gives a presentation of Γ which we now simplify.

Proposition 3.3. *A presentation of Γ is given by the generators u , v and b and the relations*

$$u^3 = v^4 = b^3 = 1, (uv)^2 = (vu)^2, vb = bv, (buv)^3 = (buvu)^2v = 1. \quad (3.5)$$

Proof. By Proposition 2.1, the set $S_0^* = \{g \in \Gamma : d(g, 0, 0) < 2r_0\}$ of generators, and the relations $g_i g_j g_k = 1$, where $g_1, g_2, g_3 \in S_0^*$, give a presentation of Γ . By Theorem 2.1, S_0^* is the union of the two double cosets $K\gamma_1 K = K$ and $K\gamma_2 K$. So these relations have the form $(k'_1 \gamma_{i_1} k''_1)(k'_2 \gamma_{i_2} k''_2)(k'_3 \gamma_{i_3} k''_3) = 1$, where $k'_\nu, k''_\nu \in K$ and $i_1, i_2, i_3 \in \{1, 2\}$. Using the known presentation of K , and cyclic permutations, the relations of the form $\gamma_{i_1} k_1 \gamma_{i_2} k_2 \gamma_{i_3} k_3 = 1$, where $1 \leq i_1 \leq i_2, i_3 \leq 2$, are sufficient to give a presentation. After finding a word in u and v for each such k_i , we obtained a list of words in u , v and b coming from these relations. Magma's routine `Simplify` can be used to complete the proof, but it is not hard to see this more directly as follows. When $i_1 = 1$, we need only include in the list of words all those coming from the relations of the form (i): $bkb = k'$, for $k, k' \in K$. When $i_1 = 2$ (so that $i_2 = i_3 = 2$ too), we need only include, for each $k_1 \in K$ such that $bk_1 b \in KbK$, a single pair (k_2, k_3) such that $bk_1 bk_2 bk_3 = 1$, since the other such relations follow from this one and the relations (i). The only relations of the form (i) are the relations $bv^\nu uvub = v^{\nu-1} u^{-1} v^{-1} u^{-1}$, $\nu = 0, 1, 2, 3$, which follow from the relations $vb = bv$ and $(buvu)^2 v = 1$ in (3.5). Next, matrix calculations show that there are 40 elements $k \in K$ such that $bkb \in KbK$, giving 40 relations $bkb = k'bk''$. We need only show that all of these are deducible from the relations given in (3.5). Using $bv = vb$, any equation $bkb = k'bk''$ gives equations $bv^i k v^j b = v^i k' b k'' v^j$. So we needed only deduce from (3.5) the five relations

$$\begin{aligned} b1b &= uvubvuvu, & bub &= ubu, & buvu^{-1}b &= v^{-1}u^{-1}v^{-1}bu^{-1}, \\ buvub &= (vu)^{-2}, & \text{and} & & buvu^{-1}vub &= v^2u^{-1}v^{-1}u^{-1}bu^{-1}v^{-1}u^{-1}. \end{aligned}$$

Firstly, $(buvu)^2 v = 1$ and $(vu)^2 = (uv)^2$ imply that $b^{-1} = uvubvuvu$, and this and $b^3 = 1$ imply the first relation. The relations $(buv)^3 = 1$ and $bv = vb$ imply that $bub = (vubvuv)^{-1}$, and this and $b^{-1} = uvubvuvu$ give $bub = ubu$. To get the third relation, use $vb = bv$ to see that $v(buvu^{-1}b)u$ equals

$$b(vuvu)ubu = b(vu)^2 bub = b(vu)^2 b(vu)^2 (vu)^{-2} ub = v(uv)^{-2} u = u^{-1} v^{-1} b.$$

The fourth relation is immediate from $(buvu)^2 v = 1$ and $bv = vb$. Finally, from $(uv)^2 = (vu)^2$ and our formula for b^{-1} we have

$$(uvu)^{-1} b^{-1} (uvu) = (uvu)^{-1} (uvubvuvu) (uvu) = vbuvu^2 vu = vbk$$

for $k = uvu^{-1}vu$, using $u^3 = 1$. So vbk has order 3. Hence $bkb = v^{-1}(kvbv)^{-1}v^{-1}$, and the fifth relation easily follows. \square

As mentioned above, $(bu^{-1})^4 = 1$. By Proposition 3.3, this is a consequence of the relations in (3.5). Explicitly,

$$\begin{aligned} (bu^{-1})^4 &= bu(ubu)(ubu)(ubu)u = bu(bub)(bub)(bub)u = b(ubu)bbubb(ubu) \\ &= b(bub)bbubb(bub) \\ &= bbuuub \\ &= 1. \end{aligned}$$

Let us record here the connection between Γ and a Deligne-Mostow group whose presentation (see Parker [3]) is

$$\Gamma_{3,4} = \langle J, R_1, A_1 : J^3 = R_1^3 = A_1^4 = 1, A_1 = (JR_1^{-1}J)^2, A_1 R_1 = R_1 A_1 \rangle.$$

Using the fact that the orbifold Euler characteristics of $\Gamma_{3,4} \backslash B(\mathbb{C}^2)$ and $\Gamma \backslash B(\mathbb{C}^2)$ are both equal to $1/288$, several experts, including John Parker, Sai-Kee Yeung and Martin Deraux, knew that Γ and $\Gamma_{3,4}$ are isomorphic. The following proof based on

presentations is a slight modification of one communicated to us by John Parker. It influenced our choice of the generators u and v for K .

Proposition 3.4. *There is an isomorphism $\psi : \Gamma \rightarrow \Gamma_{3,4}$ such that*

$$\psi(u) = JR_1J^{-1}, \quad \psi(v) = A_1, \quad \text{and} \quad \psi(b) = R_1. \quad (3.6)$$

Its inverse satisfies $\psi^{-1}(J) = buv$, $\psi^{-1}(R_1) = b$ and $\psi^{-1}(A_1) = v$.

Proof. Setting $R_2 = JR_1J^{-1}$, we have $R_1R_2A_1 = J$ and $A_1JR_2 = JR_1^{-1}J$. So

$$(\psi(u)\psi(v))^2 = (R_1^{-1}J)^2 = R_1^{-1} \cdot A_1JR_2 = A_1R_1^{-1}JR_2 = A_1R_2A_1R_2 = (\psi(v)\psi(u))^2.$$

Next, $\psi(b)\psi(u)\psi(v) = R_1R_2A_1 = J$, which implies that $(\psi(b)\psi(u)\psi(v))^3 = 1$. Now $(\psi(b)\psi(u)\psi(v)\psi(u))^2\psi(v) = (R_1R_2A_1R_2)^2A_1 = (JR_2)^2A_1 = JR_2JR_1^{-1}J = 1$. So there is a homomorphism $\psi : \Gamma \rightarrow \Gamma_{3,4}$ satisfying (3.6). We similarly check that we have a homomorphism $\tilde{\psi} : \Gamma_{3,4} \rightarrow \Gamma$ mapping J , R_1 and A_1 to buv , b and v , respectively, and that ψ and $\tilde{\psi}$ are mutually inverse. \square

We now exhibit a torsion-free subgroup of Γ having index 864. It has three generators, all in KbK . The elements of K are most neatly expressed if we use not only the generators u and v , but also $j = (uv)^2$, which is the diagonal matrix with diagonal entries ζ , ζ and 1, and which generates the center of K .

Lemma 3.2. *The non-trivial elements of finite order in Γ have order dividing 24.*

- (i) *Any element of order 2 is conjugate to one of v^2 , j^6 or $(bu^{-1})^2$.*
- (ii) *Any element of order 3 is conjugate to one of u , j^4 , uj^4 , buv , or their inverses.*

Proof. By [1, Corollary II.2.8(1)] for example, any $g \in U(2, 1)$ of finite order fixes at least one point of $B(\mathbb{C}^2)$, and so in particular this holds for any $g \in \Gamma$ of finite order. Conjugating g , we may assume that the fixed point is in the fundamental domain \mathcal{F} of Γ , and so $d(g.0, 0) \leq 2r_0$. Thus g lies in $K \cup K\gamma_2K \cup K\gamma_3K$, and so is conjugate to an element of $K \cup \gamma_2K \cup \gamma_3K$. Checking these 864 elements, we see that g must have order dividing 24. After listing the elements of order 2 and 3 amongst them, routine calculations verify (i) and (ii). \square

Proposition 3.5. *The elements*

$$a_1 = vuv^{-1}j^4buvj^2, \quad a_2 = v^2ubuv^{-1}uv^2j \quad \text{and} \quad a_3 = u^{-1}v^2uj^9bv^{-1}uv^{-1}j^8$$

generate a torsion-free subgroup Π of 864, for which $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$.

Proof. Using our presentation of Γ , Magma's `Index` command verifies that Π has index 864 in Γ , and the `AbelianQuotientInvariants` command verifies that it has abelianization \mathbb{Z}^2 .

We now check that Π is torsion-free. Suppose that Π contains an element $\pi \neq 1$ of finite order. By Lemma 3.2, we can assume that π has order 2 or 3. So for one of the elements t listed in (i) and (ii) of Lemma 3.2, there is a $g \in \Gamma$ such that $gtg^{-1} \in \Pi$. Using `Index`, one verifies that the 864 elements $b^\mu k$, $\mu = 0, 1, 2$, $k \in K$, form a transversal for Π in Γ . So we may assume that $g = b^\mu k$. But now `Index` verifies that none of the elements $b^\mu kt(b^\mu k)^{-1}$ is in Γ . \square

We conclude this section by mentioning some other properties of Π .

Let us first note that Π cannot be lifted to a subgroup of $SU(2, 1)$. The determinants of a_1 , a_2 and a_3 are ζ^3 , ζ^3 and -1 , respectively, and so the a_ν could be replaced by $\zeta^{-1}\omega^{i_1}a_1$, $\zeta^{-1}\omega^{i_2}a_2$ and $-\omega^{i_3}a_3$, where $\omega = e^{2\pi i/3}$ and $i_1, i_2, i_3 \in \mathbb{Z}$, to obtain generators with determinant 1. But $a_2^{-3}a_3a_1a_2a_3^{-3}a_2^3a_3^{-1}a_1^{-1}a_2^{-1}a_1a_3a_1^{-1} = \zeta^{-4}I$ is unchanged by any choice of the integers i_1 , i_2 and i_3 , as the number of a_ν 's appearing in the product on the left is equal to the number of a_ν^{-1} 's, for each ν . So we get

a relation in $PU(2, 1)$ but not in $SU(2, 1)$. It was found using Magma's **Rewrite** command, which derives a presentation of Π from that of $\bar{\Gamma}$.

Magma shows that the normalizer of Π in Γ contains Π as a subgroup of index 3, and is generated by Π and j^4 . One may verify that

$$\begin{aligned} j^4 a_1 j^{-4} &= \zeta^3 a_3 a_2^{-3} a_3^3 a_1, \\ j^4 a_2 j^{-4} &= \zeta^{-1} a_3^{-1}, \quad \text{and} \\ j^4 a_3 j^{-4} &= \zeta^{-1} a_1^{-1} a_2^{-1} a_1 a_2^2 a_1^{-1} a_2^{-1} a_1 a_3^{-1} a_1^{-1} a_2 a_1. \end{aligned}$$

Let us show that j^4 induces a non-trivial action on $\Pi/[\Pi, \Pi]$. By Proposition 3.5, there is an isomorphism $\varphi : \Pi/[\Pi, \Pi] \rightarrow \mathbb{Z}^2$, so we have a surjective homomorphism $f : \Pi \rightarrow \Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$. Using the relation $a_2^2 a_1^{-1} a_2^{-1} a_1 a_3^3 a_1 a_2^{-3} a_3^3 a_1 a_3 a_1 = \zeta^3 I$, we see that $3f(a_1) - 2f(a_2) + 7f(a_3) = (0, 0)$, and since $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$, this must be the only condition on the $f(a_\nu)$. So we can choose the isomorphism φ so that f maps a_1, a_2 and a_3 to $(1, 3)$, $(-2, 1)$ and $(-1, -1)$, respectively, and then

$$f(\pi) = (m, n) \implies f(j^4 \pi j^{-4}) = (m, n) \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{for all } \pi \in \Pi. \quad (3.7)$$

Next consider the ball quotient $X = \Pi \backslash B(\mathbb{C}^2)$. Now j^4 induces an automorphism of X . Let us show that this automorphism has precisely 9 fixed points.

Proposition 3.6. *The automorphism of X induced by j^4 has exactly 9 fixed points. These are the three points $\Pi(b^\mu \cdot 0)$, $\mu = 0, 1, -1$, and six points $\Pi(h_i \cdot z_0)$, where $h_i \in \Gamma$ for $i = 1, \dots, 6$, and where $z_0 \in B(\mathbb{C}^2)$ is the unique fixed point of buv .*

Proof. If $\Pi(j^4 \cdot z) = \Pi z$, then $\pi j^4 \cdot z = z$ for some $\pi \in \Pi$. This implies that πj^4 has finite order. It cannot be trivial, since Π is torsion-free. If $\pi \in \Pi$, then $\pi' = (\pi j^4)^3 = (\pi)(j^4 \pi j^8)(j^8 \pi j^4)$ is also in Π . Since the possible orders of the elements of Γ are the divisors of 24, if πj^4 has finite order, then $1 = (\pi j^4)^{24} = (\pi')^8$, so π' must be 1, so that $(\pi j^4)^3$ must be 1. So πj^4 must have order 3. So for one of the eight elements t listed in Lemma 3.2(ii), $\pi j^4 = gtg^{-1}$ for some $g \in \Gamma$. Thus $gtg^{-1}j^{-4} \in \Pi$. Since the elements $b^\mu k$, $\mu = 0, 1, -1$ and $k \in K$, form a set of coset representatives for Π in Γ , and since j^4 normalizes Π , we can assume that $g = b^\mu k$ for some μ and k .

When $t = j^4$, we have $b^\mu k t k^{-1} b^{-\mu} j^{-4} = b^\mu j^4 b^{-\mu} j^{-4}$, independent of k . We find that these three elements are in Π . Explicitly, $b^\mu j^4 b^{-\mu} j^{-4} = \pi_\mu$ for

$$\pi_0 = 1, \quad \pi_1 = \zeta^{-4} a_2 a_1^{-2} a_3^{-3} a_1^{-1} \text{ and } \pi_{-1} = a_2^2 a_1 a_3 a_1^{-1}. \quad (3.8)$$

and these equations mean that the three points $\Pi(b^\mu \cdot 0)$ are fixed by j^4 .

Write $\pi j^4 = gtg^{-1}$ for some $g \in \Gamma$, where t is one of the eight elements t listed in Lemma 3.2(ii). In the notation of (3.8), and writing $g = \pi' b^\mu k$, where $\pi' \in \Pi$, $\mu \in \{0, 1, -1\}$, and $k \in K$, we get

$$\begin{aligned} \pi j^4 &= \pi' b^\mu k t k^{-1} b^{-\mu} \pi'^{-1} = \pi' b^\mu k t k^{-1} (j^{-4} b^{-\mu} \pi_\mu j^4) \pi'^{-1} \\ &= \pi' (b^\mu k) (t j^{-4}) (b^\mu k)^{-1} (\pi_\mu j^4 \pi'^{-1} j^{-4}) j^4. \end{aligned}$$

So $(b^\mu k)(t j^{-4})(b^\mu k)^{-1}$ is in Π , and therefore either $t = j^4$ or $t j^{-4}$ has infinite order. In particular, apart from $t = j^4$, our t cannot be in K , and so must be buv or $(buv)^{-1}$.

We find that $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$ never occurs when $t = (buv)^{-1}$. For $t = buv$, we find that $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$ for only 18 pairs (μ, k) . This means that j^4 fixes $\Pi(b^\mu k \cdot z_0)$ for these 18 (μ, k) 's. If (μ, k) satisfies $b^\mu k t k^{-1} b^{-\mu} j^{-4} \in \Pi$, then so does $(\mu, k j^4)$, since we can write $b^\mu j^4 = \pi_\mu j^4 b^\mu$ for some $\pi_\mu \in \Pi$, as we have just seen. Moreover, $\Pi(b^\mu k j^4 \cdot z_0) = \Pi(b^\mu k \cdot z_0)$, since $k j^4 = j^4 k$ and so

$$\Pi(b^\mu k j^4 \cdot z_0) = \Pi(\pi_\mu j^4 b^\mu k \cdot z_0) = \Pi(j^4 b^\mu k \cdot z_0) = \Pi(b^\mu k \cdot z_0).$$

So we need only consider six of the (μ, k) 's, and correspondingly setting

$$\begin{aligned} h_1 &= b^{-1}vuj^3, & h_2 &= u^{-1}vj, & h_3 &= buv^2j^2, \\ h_4 &= b^{-1}v^2uj^3, & h_5 &= vj^2, & h_6 &= bvu^{-1}v, \end{aligned}$$

we have $h_i(buv)h_i^{-1}j^{-4} = \pi'_i \in \Pi$ for $i = 1, \dots, 6$; explicitly,

$$\begin{aligned} \pi'_1 &= \zeta^4 a_2^2 a_1 a_3^3, & \pi'_2 &= j^8 a_1 j^4, & \pi'_3 &= \zeta^2 j^8 a_1 a_2^3 j^4 a_2 a_1 a_2^{-2} a_1^{-1}, \\ \pi'_4 &= \zeta^{-5} a_3^3 a_1^2 a_3^3, & \pi'_5 &= \zeta^{-1} j^4 a_1^{-1} a_2^{-1} j^8, & \pi'_6 &= \zeta a_2 a_1^{-1}. \end{aligned}$$

The six points $\Pi(h_i.z_0)$ are distinct, as we see by checking that (a) the nontrivial $g \in \Gamma$ fixing z_0 are just $(buv)^{\pm 1}$, and (b) $(b^{\mu'}k')(buv)^{\epsilon}(b^{\mu}k)^{-1}$ is not in Π for $\epsilon = 0, 1, 2$, when (μ', k') and (μ, k) in the above list of six pairs are distinct. \square

Finally, we show that Π is a congruence subgroup of Γ .

The prime 3 ramifies in $\mathbb{Q}(\zeta)$ (as does 2), and $\mathbb{F}_9 = \mathbb{Z}[\zeta]/r\mathbb{Z}[\zeta]$ is a field of order 9. Let $\rho : \mathbb{Z}[\zeta] \rightarrow \mathbb{F}_9$ be the natural map, and write i for $\rho(\zeta)$. Then $i^2 = -1$, and $\mathbb{F}_9 = \mathbb{F}_3(i)$. Applying ρ to matrix entries, we map Γ to a group of matrices over \mathbb{F}_9 , modulo $\langle i \rangle$. The image $\rho(g)$ of any $g \in M_{3 \times 3}(\mathbb{Z}[\zeta])$ unitary with respect to the F of (3.1) is unitary with respect to $\rho(F)$, and so if we conjugate by $C = \rho(\gamma_0)$, where γ_0 is as in (3.2), then $\rho'(g) = C\rho(g)C^{-1}$ is unitary in the “usual” way.

So ρ' maps Γ to the group $PU(3, \mathbb{F}_9)$ of unitary matrices with entries in $\mathbb{F}_3(i)$, modulo scalars. This map is surjective. In fact, $\rho'(\Gamma_1) = PU(3, \mathbb{F}_9)$, where Γ_1 is the normal index 3 subgroup of Γ consisting of the $gZ \in \Gamma$ having a matrix representative g of determinant 1. One may check that $\Gamma_1 = \langle v, bu^{-1}, u^{-1}b \rangle$, and that $\langle \rho'(v), \rho'(bu^{-1}), \rho'(u^{-1}b) \rangle = PU(3, \mathbb{F}_9)$.

The given generators a_1, a_2 and a_3 of Π have determinants ζ^3, ζ^3 and -1 , respectively, and so $\Pi \subset \Gamma_1$. Now $-\zeta a_2$ and $-a_1 a_2$ are mapped by ρ' to the matrices

$$R = \begin{pmatrix} -i & -i-1 & i \\ 1 & i-1 & -1 \\ i-1 & 0 & i-1 \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} i & -i & i+1 \\ -i-1 & i & -i \\ i & -i-1 & i \end{pmatrix},$$

respectively, which satisfy $R^7 = I$, $M^3 = I$ and $MRM^{-1} = R^2$. Moreover, $-a_3$ is mapped to R^{-1} . Hence Π is mapped onto the subgroup $\langle R, M \rangle$ of $PU(3, \mathbb{F}_9)$, which has order 21. Now $|PU(3, \mathbb{F}_9)| = 6048 = 288 \times 21$, and so the conditions on a $gZ \in \Gamma$ to be in Π are that $gZ \in \Gamma_1$ and that $\rho'(g) \in \langle R, M \rangle$.

4. CALCULATION OF r_0 .

For any symmetric set $S \subset U(2, 1)$, the following lemma simplifies the description of the set \mathcal{F}_S defined in (2.1) in the case $X = B(\mathbb{C}^2)$.

Lemma 4.1. *If $g \in U(2, 1)$ and $z = (z_1, z_2) \in B(\mathbb{C}^2)$, then $d(0, z) \leq d(0, g.z)$ if and only if $|g_{3,1}z_1 + g_{3,2}z_2 + g_{3,3}| \geq 1$.*

Proof. Since $U(2, 1)$ acts transitively on $B(\mathbb{C}^2)$, we may write $z = h.0$ for some $h \in U(2, 1)$. So by (3.4), $d(0, z) \leq d(0, g.z)$ if and only if $|h_{3,3}| \leq |(gh)_{3,3}|$. Since $z_\nu = h_{\nu,3}/h_{3,3}$ for $\nu = 1, 2$, we have $(gh)_{3,3} = (g_{3,1}z_1 + g_{3,2}z_2 + g_{3,3})h_{3,3}$, and the result follows. \square

Now let Γ and $S = K \cup KbK \cup Kbu^{-1}bK \subset \Gamma$ be as in Section 3. Write r for $+\sqrt{3}$. For $1 < \rho < (r+1)\sqrt{2}$, let U_ρ denote the union of the 12 open discs in \mathbb{C} of radius 1 with centers $\rho\zeta^\lambda$, $\lambda = 0, 1, \dots, 11$. Let B_ρ denote the bounded component of $\mathbb{C} \setminus U_\rho$. The conditions on ρ ensure that B_ρ exists. See the diagram below.

Let B_1 and B_2 denote B_ρ for $\rho = (r+1)/\sqrt{2}$ and $\rho = r+1$, respectively. In the diagram, ρ' and ρ'' are the two solutions $t > 0$ of $|te^{i\pi/12} - \rho| = 1$. When

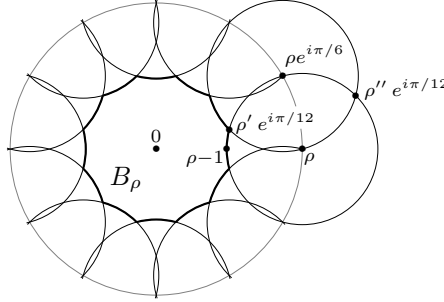
$\rho = (r+1)/\sqrt{2}$, we have $\rho' = 1$ and $\rho'' = r+1$. When $\rho = r+1$, we have $\rho' = (r+1)/\sqrt{2}$ and $\rho'' = r(r+1)/\sqrt{2}$.

Write κ for the square root of $r-1$.

Lemma 4.2. *Let $(w_1, w_2) \in \mathbb{C}^2$. Then $(w_1, w_2) \in \mathcal{F}_S$ if and only if*

- (i) $u_1 w_1 + u_2 w_2 \in B_1$ for each of the pairs $(u_1, u_2) = (\sqrt{r+1}, 0), (0, \sqrt{r+1})$ and $(\kappa^{-1}e^{-i\pi/12}, \kappa^{-1}\zeta^{3\nu}e^{-i\pi/12})$ for $\nu = 0, 1, 2, 3$, and
- (ii) $u_1 w_1 + u_2 w_2 \in B_2$ for each of the pairs $(u_1, u_2) = (\kappa^{-1}, \kappa^{-1}(\zeta+1)\zeta^{1+3\nu})$ and $(\kappa^{-1}(\zeta+1)\zeta^{1+3\nu}, \kappa^{-1})$ for $\nu = 0, 1, 2, 3$,

in which case, $|w_1|, |w_2| \leq 1/\sqrt{r+1}$.



Proof. Given $w = (w_1, w_2) \in B(\mathbb{C}^2)$, to verify that $w \in \mathcal{F}_S$, we must show that $d(0, w) \leq d(0, (bk).w)$ and that $d(0, w) \leq d(0, (bu^{-1}bk).w)$ for all $k \in K$. Since b commutes with v , and $bu^{-1}b$ commutes with u , we must check $288/4 + 288/3 = 168$ conditions.

Let γ_0 be as in (3.2), and let D , as before, be the diagonal matrix with diagonal entries 1, 1 and κ . The $g \in U(2, 1)$ to which we apply Lemma 4.1 are the matrices $(bk)^{\sim} = D\gamma_0 bk \gamma_0^{-1} D^{-1}$ and $(bu^{-1}bk)^{\sim} = D\gamma_0 bu^{-1}bk \gamma_0^{-1} D^{-1}$, where $k \in K$. Now

$$(bk)^{\sim}_{3i} = \kappa(\gamma_0 bk \gamma_0^{-1})_{3i} \quad \text{for } i = 1, 2, \quad \text{and} \quad (bk)^{\sim}_{33} = (\gamma_0 bk \gamma_0^{-1})_{33},$$

and similarly with b replaced by $bu^{-1}b$. Note also that for $\lambda \in \mathbb{Z}$,

$$(\gamma_0 bk j^{\lambda} \gamma_0^{-1})_{3i} = (\gamma_0 bk \gamma_0^{-1})_{3i} \zeta^{\lambda} \quad \text{for } i = 1, 2, \quad \text{and} \quad (\gamma_0 bk j^{\lambda} \gamma_0^{-1})_{33} = (\gamma_0 bk \gamma_0^{-1})_{33},$$

and similarly with b replaced by $bu^{-1}b$. So the conditions for $w = (w_1, w_2) \in \mathcal{F}_S$ to hold have the form

$$|\kappa(g_{31}w_1 + g_{32}w_2)\zeta^{\lambda} + g_{33}| \geq 1 \quad \text{for } \lambda = 0, \dots, 11,$$

for 6 matrices g of the form $\gamma_0 bk \gamma_0^{-1}$, and 8 matrices g of the form $\gamma_0 bu^{-1}bk \gamma_0^{-1}$. If $g = \gamma_0 bk \gamma_0^{-1}$, then $g_{33} = (\zeta+1)/\zeta$, and if $g = \gamma_0 bu^{-1}bk \gamma_0^{-1}$, then $g_{33} = r+1$.

By taking $k = uv^2u^{-1}j, j^{-2}$, and $uv^{2-\nu}j^{3(\nu-1)}$, for $\nu = 0, 1, 2, 3$, respectively, we get from $g = \gamma_0 bk \gamma_0^{-1}$ the triples (g_{31}, g_{32}, g_{33}) equal to $((\zeta+1)\zeta^{-1}, 0, (\zeta+1)\zeta^{-1})$, $(0, (\zeta+1)\zeta^{-1}, (\zeta+1)\zeta^{-1})$ and $((r+1)\zeta^{-1}/2, (r+1)\zeta^{-1}\zeta^{3\nu}/2, (\zeta+1)\zeta^{-1})$. Using $\zeta+1 = \frac{r+1}{\sqrt{2}}e^{i\pi/12}$ and $\kappa(r+1)/2 = \kappa^{-1}$, and replacing λ by $6-\lambda$, we see that the conditions coming from the six g of the form $\gamma_0 bk \gamma_0^{-1}$ are just the conditions $u_1 w_1 + u_2 w_2 \notin U_{\rho}$ for $\rho = (r+1)/\sqrt{2}$ for the six (u_1, u_2) listed in (i). Taking the case $(u_1, u_2) = (\sqrt{r+1}, 0)$, if $u_1 w_1 + u_2 w_2 = \sqrt{r+1}w_1$ is in the unbounded component of $\mathbb{C} \setminus U_{\rho}$, then $\sqrt{r+1}|w_1| \geq r+1$ (since ρ'' equals $r+1$ in this case), and so $|w_1| \geq \sqrt{r+1} > 1$, which is impossible for $(w_1, w_2) \in B(\mathbb{C}^2)$. So $\sqrt{r+1}w_1$ is in the bounded component B_{ρ} , and so $|w_1| \leq 1/\sqrt{r+1}$ (since ρ' equals 1 in this case). Similarly, $|w_2| \leq 1/\sqrt{r+1}$ for all $(w_1, w_2) \in \mathcal{F}_S$.

By taking $k = v^{1-\nu}j^{3\nu-1}$, and $k = vu^{-1}v^{2+\nu}j^9$, for $\nu = 0, 1, 2, 3$, respectively, we get from $g = \gamma_0 bu^{-1}bk \gamma_0^{-1}$ the triples (g_{31}, g_{32}, g_{33}) equal to $((r+1)/2, (r+1)(\zeta+1)\zeta^{1+3\nu}/2, r+1)$ and $((r+1)(\zeta+1)\zeta^{1+3\nu}/2, (r+1)/2, r+1)$, $\nu = 0, 1, 2, 3$. Replacing λ by $6-\lambda$, we see that the conditions coming from the eight g of the

form $\gamma_0 b u^{-1} b k \gamma_0^{-1}$ are just the conditions $u_1 w_1 + u_2 w_2 \notin U_\rho$ for $\rho = r + 1$ for the eight (u_1, u_2) listed in (ii). Using $|w_1|, |w_2| \leq 1/\sqrt{r+1}$ for $(w_1, w_2) \in \mathcal{F}_S$, we see that $u_1 w_1 + u_2 w_2$ is in the bounded component B_ρ of $\mathbb{C} \setminus U_\rho$ in each case. \square

So calculation of r_0 in this case is equivalent to calculation of the maximum value ρ_0 , say, of $|w|$ on the set of $w = (w_1, w_2) \in \mathbb{C}^2$ satisfying the conditions (i) and (ii) in Lemma 4.2, and $r_0 = \frac{1}{2} \log\left(\frac{1+\rho_0}{1-\rho_0}\right)$. As we have seen, $|w_1|, |w_2| \leq 1/\sqrt{r+1}$ for $(w_1, w_2) \in \mathcal{F}_S$. So \mathcal{F}_S is compact, and $\rho_0 \leq \sqrt{r-1}$.

We can now show that the value of r_0 is $\frac{1}{2}d(\gamma_3.0, 0) = \frac{1}{2}\cosh^{-1}(r+1)$, where $\gamma_3 = b u^{-1} b$. We first prove that this is a lower bound for r_0 .

Lemma 4.3. *For the above S , we have $r_0 \geq \frac{1}{2}d(\gamma_3.0, 0) = \frac{1}{2}\cosh^{-1}(r+1)$. That is, $\rho_0 \geq (r-1)\sqrt{r/2}$.*

Proof. Consider the geodesic $[0, \gamma_3.0]$ from 0 to $\gamma_3.0$, and let m be the point on $[0, \gamma_3.0]$ equidistant between 0 and $\gamma_3.0$. Let us show that $m \in \mathcal{F}_S$. If $m \notin \mathcal{F}_S$, there is a $g \in S$ so that $d(g.0, m) < d(0, m)$. Now $g.0 \neq 0$, so that $g \notin K$. Also,

$$d(g.0, 0) \leq d(g.0, m) + d(m, 0) < 2d(m, 0) = d(\gamma_3.0, 0),$$

and so $g \notin K\gamma_3 K$. So g must be in $K\gamma_2 K = KbK$. Since $m \in [0, \gamma_3.0]$,

$$d(g.0, \gamma_3.0) \leq d(g.0, m) + d(m, \gamma_3.0) < d(0, m) + d(m, \gamma_3.0) = d(0, \gamma_3.0),$$

so that $g^{-1}\gamma_3 \in K \cup KbK$. Now $g^{-1}\gamma_3 \notin K$, since otherwise $g.0 = \gamma_3.0$, so that m is closer to $\gamma_3.0$ than to 0. Since KbK is symmetric, we have $\gamma_3^{-1}g \in KbK$. Thus g must be in $\mathcal{G} = KbK \cap \gamma_3 KbK$. One may verify that $\mathcal{G} = (\langle u \rangle bK) \cup (\langle u \rangle b^{-1}K)$. Since u is in K and commutes with γ_3 , it fixes $[0, \gamma_3.0]$, and so $d(g.0, m)$ is constant on both double cosets $\langle u \rangle bK$ and $\langle u \rangle b^{-1}K$. Note that $u\gamma_3^{-1} = \gamma_3 u^{-1}$ is an element in Γ of order 2 which interchanges 0 and $\gamma_3.0$, and so fixes m . The map $f : g \mapsto u\gamma_3^{-1}g$ is an involution of \mathcal{G} , and $d(f(g).0, m) = d(g.0, m)$. Also, $f(b) = ub^{-1}u$, so that f interchanges the two double cosets, and so $d(g.0, m)$ is constant on \mathcal{G} . So to show the result, we need only check that $d(b.0, m) < d(0, m)$ does not hold. Now $\gamma_3.0 = (z_1, z_2)$ for $z_1 = -(r-1)\zeta^2/(2\sqrt{r-1})$ and $z_2 = (i+1)\zeta^2/(2\sqrt{r-1})$. Write $m_t = (tz_1, tz_2)$ for $0 \leq t \leq 1$. Then $m = m_t$ for $t = (1 - \sqrt{1 - |\gamma_3.0|^2})/|\gamma_3.0|^2 = (1 - \sqrt{1 - r/2})/(r/2) = r - 1$. Some routine calculations show that $b.m = m$, and so $d(b.0, m) = d(0, m)$. So $m \in \mathcal{F}_S$, and $r_0 \geq d(0, m) = \frac{1}{2}d(0, \gamma_3.0)$. \square

Proposition 4.1. *If $(w_1, w_2) \in \mathcal{F}_S$, then $|w_1|^2 + |w_2|^2 \leq 2r - 3 = \frac{r}{2}(r-1)^2$.*

Proof. Let $\mathcal{F}_{S^*} = \{z \in B(\mathbb{C}^2) : d(0, z) \leq d(g.0, z) \text{ for all } g \in S^*\}$ for $S^* = K \cup KbK$. Since $S^* \subset S$, we have $\mathcal{F}_S \subset \mathcal{F}_{S^*}$. We shall in fact show that $|w_1|^2 + |w_2|^2 \leq 2r - 3$ for $(w_1, w_2) \in \mathcal{F}_{S^*}$.

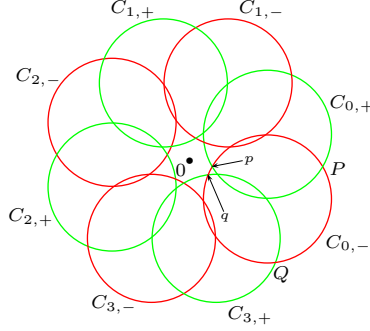
Suppose that $w = (w_1, w_2) \in \mathcal{F}_{S^*}$ and $|w_1|^2 + |w_2|^2$ is maximal. Then $d(0, w) = d(0, g.w)$ for some $g \in KbK$. Replacing w by $k.w$ for some $k \in K$, if necessary, we may suppose that $g = buv^2 u^{-1} j^7$. Let $\tilde{g} = D\gamma_0 g \gamma_0^{-1} D^{-1} \in U(2, 1)$ for this g . Since $\tilde{g}_{31} = -\kappa(\zeta + 1)\zeta^{-1}$, $\tilde{g}_{32} = 0$ and $\tilde{g}_{33} = (\zeta + 1)\zeta^{-1}$, Lemma 4.1 shows that $1 = |\tilde{g}_{31}w_1 + \tilde{g}_{32}w_2 + \tilde{g}_{33}| = |\zeta + 1| |\sqrt{r-1}w_1 - 1|$. Since $\sqrt{r+1}w_1 \in B_1$, this means that $\sqrt{r+1}w_1$ is on the rightmost arc of the boundary of B_1 . That is,

$$w_1 = \frac{1}{\sqrt{r-1}} - \frac{1}{\sqrt{r+1}} e^{-i\theta} \quad \text{for some } \theta \in [-\pi/12, \pi/12].$$

Fixing w_1 , we see from Lemma 4.2(i) that w_2 must lie on or outside various circles, including, for $\nu = 0, 1, 2, 3$ and $\epsilon = \pm$, the circles $C_{\nu, \epsilon}$ of radius $\sqrt{r-1}$ and center

$$\alpha_{\nu, \epsilon} = -i^\nu (w_1 - \sqrt{r+1} e^{i\pi/12}) = i^\nu (e^{i\pi/4} + e^{-i\theta}) / \sqrt{r+1}.$$

In the following diagram, we have taken $\theta = \pi/24$:



Let $U_\epsilon(\theta)$ denote the union of the four open discs bounded by the circles $C_{\nu,\epsilon}$, $\nu = 0, 1, 2, 3$. Using $0 < \cos(\theta + \epsilon\pi/4) < 1$, we see that $\mathbb{C} \setminus U_\epsilon(\theta)$ has two components, the bounded one containing 0. So the complement of $U(\theta) = U_+(\theta) \cup U_-(\theta)$ has a bounded component containing 0. The set $U(\theta)$ is obviously invariant under the rotations $z \mapsto i^\lambda z$, $\lambda = 0, 1, 2, 3$. It is also invariant under the reflection $R_{\nu,\epsilon}$ in the line through 0 and $\alpha_{\nu,\epsilon}$, for each ν and ϵ . For if $0 \neq \alpha \in \mathbb{C}$, the reflection in the line through 0 and α is the map $R_\alpha : z \mapsto \alpha \bar{z} / \bar{\alpha}$. It is then easy to check that

$$R_{\nu,\epsilon}(\alpha_{\nu',\epsilon'}) = \alpha_{\nu'',\epsilon'} \quad \text{for } \nu'' = 2\nu - \nu' + (\epsilon - \epsilon')/2 \pmod{4}.$$

Let p and P be the points of intersection of $C_{0,+}$ and $C_{0,-}$. Then $R_{0,-}(C_{0,+}) = C_{3,+}$ and $R_{0,-}(C_{0,-}) = C_{0,-}$, so that $R_{0,-}(p)$ and $R_{0,-}(P)$ are the points q and Q of intersection of $C_{0,-}$ and $C_{3,+}$. In particular, $|q| = |p|$ and $|Q| = |p|$.

It is now clear that $|z| \leq |p|$ for all z in the bounded component of $\mathbb{C} \setminus U(\theta)$, and that $|z| \geq |P|$ for all z in the unbounded component.

We now evaluate $|p|$ and $|P|$. Let $z \in C_{0,+} \cap C_{0,-}$. Then

$$|z + (w_1 - \sqrt{r+1}e^{i\pi/12})| = \sqrt{r-1} = |z + (w_1 - \sqrt{r+1}e^{-i\pi/12})|.$$

For $\alpha, \beta \in \mathbb{C}$ with $\beta \notin \mathbb{R}$, $|\alpha - \beta| = |\alpha - \bar{\beta}|$ if and only if $\alpha \in \mathbb{R}$. So $z + w_1$ must be real, and writing $z + w_1 = t\sqrt{r+1}$, with $t \in \mathbb{R}$, we have

$$\sqrt{r-1} = |t\sqrt{r+1} - \sqrt{r+1}e^{i\pi/12}|,$$

so that

$$t^2 - \frac{r+1}{\sqrt{2}}t + r-1 = 0.$$

The solutions of this are $t = (r-1)/\sqrt{2}$ and $t = \sqrt{2}$. Taking the smaller of these,

$$p + w_1 = \frac{r-1}{\sqrt{2}}\sqrt{r+1} = \sqrt{r-1}.$$

So $|p| = |\sqrt{r-1} - w_1|$. Taking instead $t = \sqrt{2}$, we see that $|P| = |\sqrt{2}\sqrt{r+1} - w_1|$. So $|P| \geq \sqrt{2}\sqrt{r+1} - 1/\sqrt{r+1} > 1/\sqrt{r+1}$, and therefore $|z| > 1/\sqrt{r+1}$ for all z in the unbounded component of $\mathbb{C} \setminus U(\theta)$.

So $(w_1, w_2) \in \mathcal{F}_{S^*}$ implies that w_2 is in the bounded component of $\mathbb{C} \setminus U(\theta)$, and therefore $|w_2| \leq |p| = |\sqrt{r-1} - w_1|$. Thus

$$\begin{aligned}
|w_1|^2 + |w_2|^2 &\leq |w_1|^2 + |\sqrt{r-1} - w_1|^2 \\
&= r - 1 + 2|w_1|^2 - 2\sqrt{r-1}\operatorname{Re}(w_1) \\
&= r - 1 + 2\left(\frac{1}{r-1} + \frac{1}{r+1} - \sqrt{2}\cos\theta\right) \\
&\quad - 2\sqrt{r-1}\left(\frac{1}{\sqrt{r-1}} - \frac{1}{\sqrt{r+1}}\cos\theta\right) \\
&= 3(r-1) - r(r-1)\sqrt{2}\cos\theta \\
&\leq 3(r-1) - r(r-1)\sqrt{2}\frac{r+1}{\sqrt{8}} \\
&= 2r - 3.
\end{aligned}$$

□

We conclude by giving a direct proof that the set S generates Γ . The following lemma uses a modification of an argument shown to us by Gopal Prasad.

Lemma 4.4. *If $g \in \Gamma \setminus K$, then $d(g.0, 0) \geq d(b.0, 0) = \cosh^{-1}(\sqrt{r+2})$.*

Proof. By considering the $(3, 3)$ -entry of $g^*Fg - F = 0$, we see that

$$|g_{13}|^2 + |g_{13} - (r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1). \quad (4.1)$$

Write $\alpha = g_{13}$, $\beta = g_{13} - (r-1)g_{23}$ and $\gamma = g_{33}$. By hypothesis, $g.0 \neq 0$, and so $g_{13} \neq 0$ or $g_{23} \neq 0$. Hence

$$\alpha, \beta, \gamma \in \mathbb{Z}[\zeta], \quad |\alpha|^2 + |\beta|^2 = (r-1)(|\gamma|^2 - 1), \quad \text{and } \alpha, \beta \text{ are not both 0.} \quad (4.2)$$

We claim that under conditions (4.2), $|\gamma|^2 \geq r+2$ must hold. Writing $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, we have $|\alpha|^2 = P(\alpha) + rQ(\alpha)$, where $P(\alpha) = a_0^2 + a_0a_2 + a_2^2 + a_1^2 + a_1a_3 + a_3^2$ and $Q(\alpha) = a_0a_1 + a_1a_2 + a_2a_3$. Our hypothesis is that

$$P(\alpha) + P(\beta) + P(\gamma) = 3Q(\gamma) + 1 \quad \text{and} \quad P(\gamma) = Q(\alpha) + Q(\beta) + Q(\gamma) + 1, \quad (4.3)$$

and we want to show that $P(\gamma) + rQ(\gamma) \geq 2 + r$. Now P is a positive definite form, and $\gamma \neq 0$ and either $\alpha \neq 0$ or $\beta \neq 0$. So the left hand side of the first equation in (4.3) is at least 2, so that $Q(\gamma) > 0$. Since $Q(\gamma) \in \mathbb{Z}$, we have $Q(\gamma) \geq 1$. So all we have to do is show that $P(\gamma) \geq 2$. Now $Q(\gamma) \neq 0$ implies that a_0a_1, a_1a_2 or $a_2a_3 \neq 0$. If $a_0a_1 \neq 0$, then $P(\alpha) = (a_2 + a_0/2)^2 + 3a_0^2/4 + (a_3 + a_1/2)^2 + 3a_1^2/4 \geq 3(a_0^2 + a_1^2)/4 \geq 3/2 > 1$, so that $P(\alpha) \geq 2$. The other two cases are similar. □

The following result implies that Γ is generated by K and b .

Lemma 4.5. *If $g \in \Gamma \setminus K$, then there exists $k \in K$ so that $d(bkg.0, 0) < d(g.0, 0)$.*

Proof. If there is no such $k \in K$, then $d(s.(g.0), 0) \geq d(g.0, 0)$ for all $s \in S^*$, and so $g.0 \in \mathcal{F}_{S^*}$, in the notation of the proof of Proposition 4.1, and so $d(g.0, 0) \leq r_0$. But by Lemma 4.4, $d(g.0, 0) \geq d(b.0, 0) = \cosh^{-1}(\sqrt{r+2})$, which is greater than $\frac{1}{2}\cosh^{-1}(r+1) = r_0$. □

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